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## Asymptotic equivalence and homeomorphism of the families of endomorphisms and surjections of the metrizable vector fibering

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**Abstract.** By means of the notion of abstract vector fibering of V.M.Millionshikov, the generalization of some theorems - classifiers playing an important role in the theory of asymptotic integration, is carried out.

**Keywords:** Metrizable vector fibering, surjection, bijection, homeomorphism.

**Classification:** 34A10

The perturbed differential equations may be classified on the basis of the notion of asymptotic equivalence [1 - 3]. Many theorems - classifiers playing an important role in the theory of asymptotic integration have appeared in mathematical literature lately. What kind of interpretation ensures such level of community that unites some of the results and liquidates numerous repetitions?

Evidently such interpretation is possible by means of the notion of abstract vector fibering of V.M.Millionshikov [4].

1. Let  $(E, p, B)$  be an abstract metrizable vector fibering, [4]. The surjection  $(E, p, B)$  is to be called the pair of mappings  $(Y, X^1)$ ,  $Y : E \rightarrow E$ ,  $X^1 : B \rightarrow B$  where  $pY = X^1p$ , and for any  $b \in B$  the restriction  $Y|_b \stackrel{\text{def}}{=} Y|_{p^{-1}(b)}$  of the mapping  $Y$  on the stratum over the point  $b$  is the surjection  $p^{-1}(b) \rightarrow p^{-1}(X^1b)$ .

Let  $M \subset \mathcal{R}$  and  $+\infty$  be the accumulation point of the set  $M$ . The mapping  $F_0$  of the set  $M$  into the set  $\{(Y, X^1)\}$  of all surjections  $(E, p, B)$  is called the family of the surjections of abstract vector fibering.

Let us suppose that there are defined families of endomorphisms, [4],  $F_1 : M \rightarrow \{(X, X)\}$  and surjections  $F_2 : M \rightarrow \{(Y, X^1)\}$  of the vector fibering  $(E, p, B)$ , where for some  $b \in B$   $p^{-1}(X_t b) \subseteq p^{-1}(X_t^{-1}b)$  and  $\|X_t z\| \leq K(t_0, z)Q(t)$  on these strata.  $X_t y$  and  $Y(t : t_0, z)$  are accordingly the family of endomorphisms  $F_1$  and the family of surjections  $F_2$  in the point  $(t, z) \in M \times p^{-1}(b)$ ,  $K(t_0, z)$  is a non-negative number,  $t \geq t_0$ ,  $Q : M \rightarrow (0, +\infty)$ ,  $X_{t_0} z = z$ ,  $Y(t : t_0, z) = y$ ,  $z \in p^{-1}(b)$ . Henceforth  $X.z$  and  $Y(\cdot : t_0, z)$  will be called the values of these families in the point  $z \in p^{-1}(b)$ . Suppose that  $\Phi_1 = \{X.z : z \in p^{-1}(b)\}$ ,  $\Phi_2 = \{Y(\cdot : t_0, z), z \in p^{-1}(b)\}$ ,  $\Phi_3$  is the set of all values of all families of surjections  $F_3$  such that  $F_3 : M \rightarrow \{(Z, X^2)\}$ ,  $p^{-1}(X_t^2 b) = p^{-1}(X_t^{-1}b)$ , and if  $Z(\cdot : t_0, z) \in \Phi_3$ , then  $z \in p^{-1}(b)$ ,  $\|Z(t : t_0, z)\| \leq K_3 Q(t)$ , where the positive number  $K_3$  depends on the family of surjections and on the point  $(t_0, z) \in M \times p^{-1}(b)$ . Obviously,  $\Phi_1 \subset \Phi_2$ .

Let us assume that the linear operations are defined into  $\Phi_3$  in natural way and

$$\|Z\|_{\Phi_3} = \sup_{t \geq t_0} \frac{\|Z(t : t_0, z)\|}{Q(t)}, \quad Z \in \Phi_3.$$

Then  $\Phi_3$  is the known Banach space.

**Definition 1.** The families  $F_1$  and  $F_2$  are called asymptotically equivalent according to Levinson on the stratum  $p^{-1}(b)$  with respect to the function  $Q$ , if there exist the bijection  $P : p^{-1}(b) \rightarrow p^{-1}(b)$  such that  $X_t z = Y(t : t_0, Pz) + o(Q(t))$  for  $t \rightarrow +\infty$  and for any  $z \in p^{-1}(b)$ . The families  $F_1$  and  $F_2$  are called homeomorphic on the stratum  $p^{-1}(b)$ , if  $\Phi_1$  and  $\Phi_2$  are homeomorphic in the topology of the space  $\Phi_3$ .

**Definition 2.** The families  $F_1$  and  $F_2$  are called asymptotically equivalent according to Nemitsky on the stratum  $p^{-1}(b)$  with respect to the function  $Q$ , if they are asymptotically equivalent according to Levinson on this stratum and  $P$  is a homeomorphism.

**Theorem 1.** Let us assume that

$$(1) \quad Y(t : t_0, z) = X_t z + X_t T_t Y(\cdot : t_0, z)$$

and, on the contrary, if the surjection  $\varphi(t)$  satisfies the equation (1), then  $\varphi(t) \equiv Y(t : t_0, z)$  for  $t \geq t_0$  and  $\forall z \in p^{-1}(b)$ , where:

- a)  $T_t : \Phi_3 \rightarrow p^{-1}(b)$  for  $t \geq t_0$ ,
- b)  $T_t Z(\cdot : t_0, z) = \alpha + H_t Z(\cdot : t_0, z)$ ,  $\alpha \in p^{-1}(b)$  and depends on  $Z(\cdot : t_0, z) \in \Phi_3$ ,  $H_t 0 = 0$ ,
- c)  $\|H_t Z_1(\cdot : t_0, z_1) - H_t Z_2(\cdot : t_0, z_2)\| \leq \|g_t\| \|Z_1(\cdot : t_0, z_1) - Z_2(\cdot : t_0, z_2)\|$  for  $\forall z_1, z_2 \in p^{-1}(b)$ ,  $g_t \in p^{-1}(b)$  for any  $t \geq t_0$ ,  $\|g_t\| \rightarrow 0$  for  $t \rightarrow +\infty$

Then the families  $F_1$  and  $F_2$  are asymptotically equivalent according to Levinson on the stratum  $p^{-1}(b)$  with respect to the function  $Q$ .

**PROOF :** We consider the operator

$$(2) \quad L : \Phi_3 \rightarrow \Phi_3$$

where

$$LZ_2(\cdot : t_0, z_2) = Z_1(\cdot : t_0, z_1) \\ Z_1(t : t_0, z_1) = X_t z + X_t H_t Z_2(\cdot : t_0, z_2).$$

Then  $L$  is the contracting operator in  $\Phi_3$  and, consequently, for any  $y \in p^{-1}(b)$  there exists a fixed point of  $L$  in  $\Phi_3$ , that is

$$Z(t : t_0, z_0) = X_t z + X_t H_t Z(\cdot : t_0, z_0), \quad Z(\cdot : t_0, z_0) \in \Phi_3.$$

Then on the basis of b) we have

$$Z(t : t_0, z_0) = X z_0 + X_t T_t Z(\cdot : t_0, z_0), \quad z_0 \in p^{-1}(b).$$

Taking into consideration (1), hence, it follows  $Z(\cdot : t_0, z_0) \in \Phi_2$ . Here  $Pz = z_0$ , where  $P$  is the bijection  $p^{-1}(b)$  on  $p^{-1}(b)$ ,  $T_t Z(\cdot : t_0, z + \alpha) = \alpha + H_t Z(\cdot : t_0, z + \alpha)$  and  $Z(t : t_0, Pz) = X_t z + o(Q(t))$  for  $t \rightarrow +\infty$ . ■

**Lemma 2.** Let  $D$  be a Banach space and  $f : D \rightarrow D$  be a contracting operator,  $U$  and  $V$  non-empty subsets of  $D$  such that  $(I - f)V \subset U$  ( $I$  is an identity operator). If  $S : U \rightarrow V$  satisfies the correlation

$$Sy = y + fSy$$

then  $S$  is the homeomorphism of  $U$  on  $V$ .

PROOF : Since  $f$  is a contracting operator, then for any  $y$   $Sy$  is a unique solution of the equation

$$(3) \quad Sy = y + fSy, \quad y \in V$$

Suppose that  $y_1 \neq y_2$ . Let us show that  $Sy_1 \neq Sy_2$ . If we assume the contrary, then

$$Sy_1 = y_1 + fSy_1, \quad Sy_2 = y_2 + fSy_2$$

and  $y_1 - y_2 = 0$ . Hence,  $Sy_2 \neq Sy_1$ . Let  $x$  be a fixed element. Let us consider the equation

$$(4) \quad x = y + fx$$

In this case  $y = x - fx$ , that is the equation (4) has the unique solution. Hence, there exists  $S^{-1}$ . Let us prove the continuity of  $S$  and  $S^{-1}$ . Let  $y_n \rightarrow y_0$  for  $n \rightarrow +\infty$ . Then

$$Sy_n - Sy_0 = y_n - y_0 + fSy_n - fSy_0$$

and

$$\|Sy_n - Sy_0\| \leq \|y_n - y_0\| + q\|Sy_n - Sy_0\|, \quad 0 < q < 1.$$

Hence, it follows

$$\|Sy_n - Sy_0\| \leq \frac{1}{1-q} \|y_n - y_0\|$$

that is

$$Sy_n \rightarrow Sy_0 \quad \text{for } y_n \rightarrow y_0.$$

Let us prove the continuity of  $S^{-1}$ . It is easy to notice that  $S^{-1}y$  satisfies the correlation

$$y = S^{-1}y + fy, \quad y \in V.$$

Then, if  $y_n \rightarrow y_0$  for  $n \rightarrow +\infty$ , then

$$\|S^{-1}y_n - S^{-1}y_0\| \leq \|y_n - y_0\| + q\|y_n - y_0\|$$

that is  $S^{-1}y_n \rightarrow S^{-1}y_0$ . The lemma is proved.

**Theorem 2.** Let us assume that the conditions a) and b) of Theorem 1 are fulfilled. If

$$\|H_t Z_1(\cdot : t_0, z_1) - H_t Z_2(\cdot : t_0, z_2)\| \leq q \|Z_1(\cdot : t_0, z_1) - Z_2(\cdot : t_0, z_2)\|_{\Phi_3}, \quad 0 < q < 1$$

then the sets  $\Phi_1$  and  $\Phi_2$  are homeomorphic in the topology of the space  $\Phi_3$ .

PROOF : On the basis of (2) we consider the operator

$$(5) \quad \begin{aligned} L_X Z &= X + FZ, \quad X = X.z \in \Phi_1, \quad Z \in \Phi_3, \quad FZ = Z_1 \\ Z_1(t : t_0, z_1) &= X_t H_t Z(\cdot : t_0, z) \end{aligned}$$

$F$  is a contracting operator here. We define the operator  $S : \Phi_1 \rightarrow \Phi_2$  as follows: for any  $X \in \Phi_1$   $SX$  is the fixed point of the contracting operator  $L_X$ . Then

$$SX = L_X SX$$

If we assume that  $U = \Phi_1$ ,  $V = \Phi_2$ , then, on the basis of the Lemma,  $S$  is a homeomorphism of  $\Phi_1$  on  $\Phi_2$ . ■

**Corollary.** From Theorem 2 it follows that the families  $F_1$  and  $F_2$  are homeomorphic on the stratum  $p^{-1}(b)$ . Moreover, they are asymptotically equivalent according to Nemitsky on this stratum.

3. Now we consider the case, when on each stratum  $p^{-1}(X_t, b)$

$$(6) \quad \|X_t z\| \leq K_i(t_0, z) Q_i(t)$$

where

$$z \in \Xi_i, \bigcup_i \Xi = p^{-1}(b), K_i : M \times \Xi_i \rightarrow (0, +\infty), Q_i : M \rightarrow (0, +\infty), i = \overline{1, n}.$$

Suppose that  $\Phi_1^{(i)} = \{X_0 z : z \in \Xi_i\}$ ,  $\Phi_3^{(i)}$  is the set of all values of all families of surjections  $F_3^{(i)}$  such that  $F_3^{(i)} : M \rightarrow \{(Z, X^2)\}$ ,  $p^{-1}(X_t^2 b) = p^{-1}(X_t^1 b)$  and if  $Z(\cdot : t_0, z) \in \Phi_3^{(i)}$ , then  $z \in p^{-1}(b)$ ,  $\|Z(t : t_0, z)\| \leq K_3^{(i)} Q_i(t)$ , where the positive number  $K_3^{(i)}$  depends on the family of surjections and on the point  $(t_0, z) \in M \times p^{-1}(b)$ ,  $\Phi_1^{(i)} \in \Phi_3^{(i)}$ ,  $i = \overline{1, n}$ .

**Theorem 3.** Assume that

$$(7) \quad Y(t : t_0, z) = X_t z + X_t T_t^{(i)} y(\cdot : t_0, z), \quad i = \overline{1, n}$$

and, on the contrary, if  $\phi(t)$  satisfies the equation (7), then  $\phi(t) \equiv Y(t : t_0, z)$  for  $t \geq t_0$  and  $\forall z \in \Xi_i$ , where

- $T_t^{(i)} : \Phi_3^{(i)} \rightarrow p^{-1}(b)$  for  $t \rightarrow t_0$ ,
- $T_t^{(i)} Z(\cdot : t_0, z) = \alpha + H_t^{(i)} Z(\cdot : t_0, z)$ ,  $\alpha \in p^{-1}(b)$  and depends on  $Z(\cdot : t_0, z) \in \Phi_3^{(i)}$ ,  $H_t^{(i)} 0 = 0$
- $\|H_t^{(i)} Z_1(\cdot : t_0, z_1) - H_t^{(i)} Z_2(\cdot : t_0, z_2)\| \leq \|gt\| \|Z_1(\cdot : t_0, z_1) - Z_2(\cdot : t_0, z_2)\|_{\Phi_3^{(i)}}$  for  $z_1, z_2 \in p^{-1}(b)$ ,  $g_i \in p^{-1}(b)$  for any  $t \geq t_0$ ,  $\|g_i\| \rightarrow 0$  for  $t \rightarrow +\infty$ .

Then there exist the sets  $\Xi_i \subset p^{-1}(b)$ ,  $\bigcup_i \Xi_i = p^{-1}(b)$  and the bijections  $P_i: \Xi_i \rightarrow \Xi_i$  such that  $X_t z = Y(t: t_0, P_i z) + o(Q_i(t))$  for  $t \rightarrow +\infty$  and for any  $z \in \Xi_i$ ,  $i = \overline{1, n}$ .

The proof of Theorem 3 is analogous to the proof of Theorem 1.

**Theorem 4.** Let us assume that  $\Phi_2^{(i)} = \{Y(\cdot: t_0, z): z \in \Xi_i\}$  and the conditions a) and b) of Theorem 3 are fulfilled. Then, if

$$\|H_t^{(i)} Z_1(\cdot: t_0, z_1) - H_t^{(i)} Z_2(\cdot: t_0, z_2)\| \leq q_i \|Z_1(\cdot: t_0, z_1) - Z_2(\cdot: t_0, z_2)\|_{\Phi_3^{(i)}}, \quad 0 < q_i < 1, \quad i = \overline{1, n}$$

then the sets  $\Phi_1^{(i)}$  and  $\Phi_2^{(i)}$  are homeomorphic in the topology of the space  $\Phi_3^{(i)}$  for any  $1 \leq i \leq n$ .

The proof of Theorem 4 does not differ from the proof of Theorem 2.

Theorem 1 is the transference of the notion of asymptotic equivalence of the differential equations according to Levinson [3], [5], when one of them is a linear homogeneous equation, onto metrizable abstract vector fiberings. Specifically the known Levinson theorem follows from Theorem 1. Theorem 2 is a new theorem. From the Lemma it follows that in the conditions of the Levinson theorem for the considered differential equations to be fulfilled the asymptotical equivalence to Nemitsky take place.

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