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Abstract. We describe necessary and sufficient conditions for a direct product and a lexicographic product of partially ordered quasigroups to be a positive quasigroup. Analogous questions for Riesz quasigroups are studied.

Keywords: partially ordered quasigroup, positive quasigroup, Riesz quasigroup, direct product, lexicographic product

Classification: 20N05, 06F99

1. Preliminaries

The concept of an ordered loop was introduced by D. Zelinsky [10] who was the first to consider valuations of nonassociative algebras. Their values are in ordered loops and D. Zelinsky [11] determined all such ordered loops. Ordered loops and quasigroups were later studied by several other authors (e.g. [1], [2], [3], [4]), also in the connection with the ordered planar ternary rings ([5]). The previous research seems to indicate that the area is interesting and rich enough to justify a systematic study. In this paper we shall consider products of special types of ordered quasigroups — positive quasigroups and Riesz quasigroups.

The concept of a positive quasigroup was introduced by V.M. Tararin [7]. Further, properties of left-positive quasigroups and left-positive Riesz quasigroups were studied by V.A. Testov [8], [9].

Let $(Q, \cdot)$ be a quasigroup. Let $a \in Q$. By e_a (f_a) we denote the local left (right) unit element for a, i.e., e_a, f_a are such elements that $e_a a = a$ and $af_a = a$. If $(Q, \cdot)$ is a loop, we denote by 1 the unit element of $(Q, \cdot)$.

A nonempty set Q with a binary operation $\cdot$ and a relation $\leq$ is called a *partially ordered quasigroup* (*po-quasigroup*) if

(i) $(Q, \cdot)$ is a quasigroup;
(ii) $(Q, \leq)$ is a partially ordered set;
(iii) for all $x, y, a \in Q$, $x \leq y \Leftrightarrow ax \leq ay \Leftrightarrow xa \leq ya$.

A po-quasigroup Q is called a *partially ordered loop* (*po-loop*) if $(Q, \cdot)$ is a loop.

We say that a po-quasigroup Q is *trivially ordered*, if any two different elements $a, b \in Q$ are non-comparable (for non-comparable elements we will use the notation $a \parallel b$).

Let Q be a po-quasigroup. An element $p \in Q$ is called a *positive element*, if $px \geq x$ and $xp \geq x$ for all $x \in Q$. The set of all positive elements of Q will be denoted by P_Q. Obviously, $P_Q = \{p \in Q : \ p \geq e_x$ and $p \geq f_x$ for all $x \in Q\}$. A po-quasigroup Q is said to be a *positive quasigroup*, if for all $x, y \in Q$, $x < y$, there exist positive elements $p, q \in P_Q$ such that $y = px$ and $y = xq$.

Clearly, a partially ordered loop Q (and obviously a partially ordered group, too) is a positive quasigroup with the set of all positive elements $P_Q = \{p \in Q : p \geq 1\}$. If Q is a trivially ordered quasigroup, then Q is a positive quasigroup. The following example shows that there exists a non-trivially ordered positive quasigroup which is not a loop.

1.1 Example. Let $Q = \mathbb{R} \times \mathbb{R}$ ($\mathbb{R}$ is the set of all real numbers). Take the operation

$$(x, y) \cdot (u, v) = (x + u, 2y + v)$$

and the relation $\leq$, where $=$ is defined componentwise and $<$ is defined by

$$(x, y) < (u, v) \Leftrightarrow x < u.$$

Then Q is a positive quasigroup with the set of all positive elements $P_Q = \{(x, y) : x > 0\}$. The set of all local unit elements of Q is $E = \{(0, y) : \ y \in \mathbb{R}\}$.

It is easy to verify that a direct product and a lexicographic product of partially ordered quasigroups is a partially ordered quasigroup. The situation is different in the case of positive quasigroups. In this note we are interested in conditions under which a direct product and a lexicographic product of partially ordered quasigroups is a positive quasigroup. Analogous questions for Riesz quasigroups are studied as well.

2. Direct products of positive quasigroups

Let I be a nonempty set and let $\{Q_i : i \in I\}$ be a family of partially ordered quasigroups. By the *direct product* of po-quasigroups Q_i, $i \in I$, we mean the Cartesian product of sets Q_i with the operation $\cdot$ and the relation $\leq$ defined componentwise, i.e.,

$$x \cdot y = z \Leftrightarrow x(i) \cdot y(i) = z(i) \ \text{ for all } \ i \in I,$$
$$x \leq y \Leftrightarrow x(i) \leq y(i) \ \text{ for all } \ i \in I,$$

where $x(i)$, $y(i)$ and $z(i)$ is the ith component of x, y and z, respectively. The direct product of po-quasigroups Q_i, $i \in I$ will be denoted by $\prod_{i \in I} Q_i$.

The direct product of positive quasigroups need not be a positive quasigroup. For instance, take Q as in 1.1. Then the direct product $\prod_{i \in I} Q_i$, where $I = \{1, 2\}$ and $Q_i = Q$ for $i = 1, 2$, is not a positive quasigroup.

2.1 Lemma. *Let I be a nonempty set and let $\{Q_i : i \in I\}$ be a family of partially ordered quasigroups. If $Q = \prod_{i\in I} Q_i$ is a positive quasigroup, then each Q_i is a positive quasigroup, too.*

PROOF: Let $Q = \prod_{i\in I} Q_i$ and let $j \in I$. If Q_j is a trivially ordered quasigroup, then it is a positive quasigroup. Assume that Q_j is non-trivially ordered. Let $a, b \in Q_j$, $a > b$. There exist $p, q \in Q_j$ such that $a = qb = bp$. We are going to show that p, q are positive elements. Let us define $x, y \in Q$ in such a way that $x(j) = a$, $y(j) = b$ and $x(i) = y(i)$ for each $i \in I$, $i \neq j$. Clearly $x > y$. Since Q is a positive quasigroup, there exists $u, v \in P_Q$ such that $x = uy = yv$. Evidently $u(j) = q$, $v(j) = p$. Now, let c be any element of Q_j. Take $z \in Q$ with $z(j) = c$. Since $u \in P_Q$, $uz \geq z$ and $zu \geq z$. Thus we have $qc \geq c$ and $cq \geq c$. Analogously, $pc \geq c$ and $cp \geq c$. Thus $p, q \in P_{Q_j}$. □

2.2 Lemma. *Let I be a nonempty set and let $\{Q_i : i \in I\}$ be a family of partially ordered quasigroups such that there are at least two non-trivially ordered quasigroups. Then $Q = \prod_{i\in I} Q_i$ is a positive quasigroup if and only if Q_i is a po-loop for each $i \in I$.*

PROOF: Let $Q = \prod_{i\in I} Q_i$ be a positive quasigroup. Let e_a, e_b be the local left unit elements for $a, b \in Q_j$, respectively. We are going to show that $e_a = e_b$. By assumption there exists $k \in I$, $k \neq j$, such that Q_k is a non-trivially ordered quasigroup. Let us take $x, y \in Q$ such that $x(j) = y(j) = b$, $x(k) > y(k)$ and $x(i) = y(i)$ for each $i \in I - \{j, k\}$. Since $x > y$, there is $p \in P_Q$ such that $x = py$. Obviously, $p(j) = e_b$. Now, let z be an element of Q with $z(j) = a$. Since $p \in P_Q$, $pz \geq z$. Hence $p(j)z(j) \geq z(j)$, i.e., $e_b a \geq a$. This yields $e_b \geq e_a$. Analogously we can prove that $e_a \geq e_b$. Therefore $e_a = e_b$. By the similar way we obtain that any two local right unit elements from Q_j are equal. Thus we can conclude that Q_j is a po-loop.

Conversely, if Q_i is a partially ordered loop for each $i \in I$, then $Q = \prod_{i\in I} Q_i$ is a partially ordered loop, too. Thus Q is a positive quasigroup. □

2.3 Theorem. *Let I be a nonempty set and $\{Q_i : i \in I\}$ a family of partially ordered quasigroups. Then $Q = \prod_{i\in I} Q_i$ is a positive quasigroup if and only if one of the following conditions is fulfilled:*

- (i) *Q_i is a trivially ordered quasigroup for each $i \in I$;*
- (ii) *there exists an index $k \in I$ such that Q_k is a non-trivially ordered positive quasigroup and for all $i \in I$, $i \neq k$, Q_i is a trivially ordered loop;*
- (iii) *there exist at least two indices $j, k \in I$ such that Q_j, Q_k are non-trivially ordered quasigroups and Q_i is a partially ordered loop for each $i \in I$.*

PROOF: Let $Q = \prod_{i\in I} Q_i$ be a positive quasigroup. By 2.1 Q_i is a positive quasigroup for each $i \in I$. Suppose that there exists exactly one index $k \in Q$ such that Q_k is a non-trivially ordered positive quasigroup. By the same way as in the proof of 2.2 we obtain that Q_j is a loop for each $j \neq k$. Obviously Q_j is a trivially ordered loop. If there exist two indices $k, j \in I$ such that Q_k and Q_j are non-trivially ordered positive quasigroups, then, according to 2.2, Q_i is a partially ordered loop for each $i \in I$.

Conversely, from (i) it follows that $Q = \prod_{i \in I} Q_i$ is a trivially ordered quasigroup and thus it is a positive quasigroup. Let (ii) hold and let $x, y \in Q$, $x > y$. There exist $p, q \in Q$ such that $x = py = qy$. We are going to show that $p, q \in P_Q$. Obviously $x(k) > y(k)$ and $x(i) = y(i)$ for each $i \neq k$. By assumption Q_k is a positive quasigroup, therefore $p(k), q(k) \in P_{Q_k}$. Further, for each $i \in I$, $i \neq k$, $p(i) = q(i) = 1$. Thus $pz \geq z$ and $zq \geq z$ for each $z \in Q$. Hence $p, q \in P_Q$ and Q is a positive quasigroup. Finally, in view of 2.2, (iii) implies that Q is a positive quasigroup. □

2.4 Corollary. *Let Q be a partially ordered quasigroup which can be expressed as a direct product of non-trivially ordered quasigroups. Then Q is positive if and only if Q is a po-loop.*

3. Lexicographic products of positive quasigroups

Let I be a well-ordered set and let $\{Q_i : i \in I\}$ be a family of partially ordered quasigroups. By the *lexicographic product* of Q_i, $i \in I$, we mean the direct product of quasigroups Q_i with the relation $\leq$ defined by

$$x \leq y \Leftrightarrow x = y \text{ or } x(i) < y(i) \text{ for the least } i \in I \text{ with } x(i) \neq y(i),$$

where $x(i)$ and $y(i)$ is the ith component of x and y, respectively. The lexicographic product of po-quasigroups Q_i, $i \in I$, will be denoted by $\Gamma_{i \in I} Q_i$.

The lexicographic product of positive quasigroups need not be a positive quasigroup. For instance, the lexicographic product $\Gamma_{i \in I} Q_i$, where $I = \{1, 2\}$ and Q_i is a positive quasigroup from 1.1 for each $i \in I$, is not a positive quasigroup.

Using the similar methods to those in the proof of 2.1 the following lemma can be proved.

3.1 Lemma. *Let I be a well-ordered set and $\{Q_i : i \in I\}$ a family of partially ordered quasigroups. If $Q = \Gamma_{i \in I} Q_i$ is a positive quasigroup, then each Q_i is a positive quasigroup, too.*

3.2 Lemma. *Let I be a well-ordered set and $\{Q_i : i \in I\}$ a family of partially ordered quasigroups. Let $Q = \Gamma_{i \in I} Q_i$ be a positive quasigroup. If Q_k, $k \in I$, is a non-trivially ordered quasigroup, then Q_i is a po-loop for each $i \in I$, $i < k$.*

Proof: Let $j \in I$, $j < k$. Let e_a, e_b be local left unit elements for $a, b \in Q_j$, respectively. We take $x, y \in Q$ with $x(j) = y(j) = b$, $x(k) > y(k)$ and $x(i) = y(i)$ for each $i \neq j, k$. Clearly $x > y$. Thus there is $p \in P_Q$ such that $x = py$. Evidently, $p(i)$ is the local left unit for $x(i)$ for each $i \neq k$. Especially, for j we have $p(j) = e_b$. Now, let z be an element of Q with $z(j) = a$ and $z(i) = x(i)$ for each $i \neq j$. Since $p \in P_Q$, $z \leq pz$. Therefore $z(j) \leq p(j)z(j)$, i.e., $a \leq e_b a$. Hence $e_a \leq e_b$. Using analogous methods, it can be shown that $e_b \leq e_a$. Therefore $e_a = e_b$. Similarly we obtain that any two local right unit elements from Q_j are equal. □

3.3 Theorem. *Let I be a well-ordered set and $\{Q_i : i \in I\}$ a family of partially ordered quasigroups. Let each Q_i, $i \in I$, contain more than one element. Then $Q = \Gamma_{i \in I} Q_i$ is a positive quasigroup if and only if Q_i is a positive quasigroup for each $i \in I$ and*

- (i) *if Q_k, $k \in I$, is a non-trivially ordered quasigroup, then Q_j is a po-loop for each $j \in I$, $j < k$;*
- (ii) *if Q_k, $k \in I$, is a non-trivially ordered quasigroup and Q_i is a trivially ordered quasigroup for each $i > k$, then every two different local unit elements from Q_k are non-comparable.*

Proof: Let $Q = \Gamma_{i \in I} Q_i$ be a positive quasigroup. From 3.1 it follows that Q_i is a positive quasigroup for each $i \in I$. The assertion (i) follows from 3.2. To prove (ii) suppose that Q_k is a non-trivially ordered quasigroup and Q_i is a trivially ordered quasigroup for each $i > k$. First, assume that there are different local left unit elements e_a, e_b for $a, b \in Q_k$, respectively. We are going to show that $e_a \parallel e_b$. Assume that $e_a > e_b$. There exists $c \in Q_k$ such that $b = e_a c$. Since $e_a b > b$, we have $b > c$. Let x, y be elements of Q with $x(k) = b$, $y(k) = c$, $x(i) \parallel y(i)$ for $i > k$ and $x(i) = y(i)$ for $i < k$. Obviously, $x > y$, and thus $x = py$, where $p \in P_Q$. Clearly $p(k) = e_a$ and, by (i), $p(i) = 1$ for each $i < k$. Now, take $z \in Q$ with $z(k) = a$, $z(i) = y(i)$ for $i > k$ and $z(i) = x(i)$ for $i < k$. Then $p(i)z(i) = z(i)$ for $i \leq k$. Further, since $p(i)z(i) = p(i)y(i) = x(i)$ for $i > k$, we can conclude that $p(i)z(i) \parallel z(i)$ for each $i \in I$, $i > k$. Thus $pz \parallel z$. But, on the other hand, since $p \in P_Q$, we have $pz \geq z$, which contradicts the relation above. Assuming $e_b < e_a$ we again obtain a contradiction. Therefore $e_a \parallel e_b$. Analogously we can show that any two different local right units from Q_k are non-comparable. Finally, to end this direction of the proof, suppose that e_a is the local left unit for $a \in Q_k$, f_b is the local right unit for $b \in Q_k$ and $e_a \neq f_b$. Let $e_a > f_b$. Then $be_a > b$ and since Q_k is a positive quasigroup, we have $be_a = bp$, where $p \in P_{Q_k}$. Therefore $e_a \in P_{Q_k}$, i.e., $e_a \geq e_x$ for each $x \in Q_k$. Now, using the fact that any two different local left unit elements from Q_k are non-comparable, we obtain $e_a = e_x$ for each $x \in Q_k$. This yields $e_a e_a = e_a$ and therefore $e_a = f_{e_a}$, where f_{e_a} is the local right unit for e_a. And since $e_a > f_b$, we have $f_{e_a} > f_b$, which contradicts the fact that any two different local right unit elements from Q_k are non-comparable. Analogously the case $f_b > e_a$ cannot occur. Thus $e_a \parallel f_b$.

To prove the converse, let Q_i be a positive quasigroup for each $i \in I$ and let (i), (ii) be valid. A quasigroup $Q = \Gamma_{i \in I} Q_i$ is trivially ordered if and only if Q_i is a trivially ordered quasigroup for each $i \in I$. In this particular case, Q is a positive quasigroup. In the next we assume that Q is a non-trivially ordered quasigroup.

Let $x, y \in Q$, $x > y$. There are $p, q \in Q$ such that $x = py = yq$. We are going to show that $p, q \in P_Q$. Since $x > y$, there exists $k \in I$ such that $x(k) > y(k)$ and $x(i) = y(i)$ for each $i \in I$, $i < k$. Obviously $p(k) \in P_{Q_k}$ and, by (i), Q_i is a loop for each $i < k$. Moreover, for each $i < k$, $p(i)$ is the unit of Q_i. Let z be any element of Q. Clearly, $p(k)z(k) \geq z(k)$ and $p(i)z(i) = z(i)$ for $i < k$. Suppose that there exists $r \in I$, $r > k$, such that Q_r is a non-trivially ordered quasigroup. Then, by (i), Q_k is a loop. Since $x(k) = p(k)y(k)$ and $x(k) > y(k)$, we have $p(k) \neq 1$. Thus $p(k)z(k) > z(k)$, and therefore $pz > z$. Assume that Q_i is a trivially ordered quasigroup for each $i > k$. If Q_k is a loop, then again $pz > z$. Suppose that Q_k is not a loop. Since $p(k) \in P_{Q_k}$, $p(k)$ is greater than or equal to any local unit from Q_k. But, according to (ii), any two different local unit elements from Q_k are non-comparable. This yields that $p(k)$ is the local unit for none element of Q_k. Therefore $p(k)z(k) > z(k)$ and hence $pz > z$. We have shown that $p \in P_Q$. Analogously we can prove that $q \in P_Q$. Thus Q is a positive quasigroup. □

3.4 Corollary. *Let I be a well-ordered set, $\{Q_i : i \in I\}$ a family of non-trivially ordered quasigroups. Then $Q = \Gamma_{i\in I} Q_i$ is a positive quasigroup if and only if Q_i is a positive quasigroup for each $i \in I$ and for each couple (j, k) of elements of I, if k covers j, then Q_j is a po-loop.*

4. Riesz quasigroups

Riesz groups were studied by L. Fuchs, G. Birkhoff and some other authors. The necessary and sufficient conditions for a lexicographic product of a family of partially ordered group to be a Riesz group were given by J. Lihová in [6]. In this section we deal with the lexicographic product of Riesz quasigroups. As for direct products of Riesz quasigroups, it is routine to verify that the direct product of po-quasigroups is a Riesz quasigroup if and only if each factor is a Riesz quasigroup.

A partially ordered quasigroup Q is said to be a *directed quasigroup* if Q is a directed set (i.e. for each $a, b \in Q$ there exist $c, d \in Q$ such that $c \leq a, b$ and $a, b \leq d$).

Let $a, b \in Q$. By $U(a, b)$ ($L(a, b)$) we denote the set of all upper (lower, respectively) bounds of the set $\{a, b\}$.

4.1 Lemma. *Let Q be not a directed po-quasigroup. Then there exist $u, v, z \in Q$ such that $U(z, u) = \emptyset$ and $L(z, v) = \emptyset$.*

Proof: Suppose that Q is not a directed quasigroup. Then there are elements $a, b \in Q$ such that $U(a, b) = \emptyset$. In fact, if we assume that $U(x, y) \neq \emptyset$ for all $x, y \in Q$, then for each $x, y \in Q$ there exists $g \in Q$ such that $R_x^{-1}x$, $R_y^{-1}x \leq g$. Hence $L_g^{-1}x \leq x$, y and thus $L(x, y) \neq \emptyset$ for all $x, y \in Q$. This yields that Q is a directed quasigroup, which contradicts the assumption. Analogously, provided $L(x, y) \neq \emptyset$ for all $x, y \in Q$ we also arrive at contradiction. Therefore there exist $a, b, c, d \in Q$ such that $U(a, b) = \emptyset$ and $L(c, d) = \emptyset$. Then $U(ac, bc) = \emptyset$ and $L(ac, ad) = \emptyset$. Now, setting $z = ac$, $u = bc$ and $v = ad$ we obtain the required elements. □

4.2 Definition. A partially ordered quasigroup Q is called a Riesz quasigroup if it is directed and satisfies the following interpolation property

$$\text{(IP)}\qquad \begin{array}{c} \text{for all } a_i, b_j \in Q \text{ with } a_i \leq b_j,\ i, j \in \{1, 2\}, \\ \text{there exists } c \in Q \text{ such that } a_i \leq c \leq b_j. \end{array}$$

Evidently every Riesz group is a Riesz quasigroup (Riesz groups are exactly the associative Riesz quasigroups). To give an example of a Riesz quasigroup which is not a Riesz group consider $Q = \mathbb{R}^2$ with operation $(x, y) \cdot (u, v) = (x + u, \frac{1}{2}(y + v))$ and relation $(x, y) < (u, v) \Leftrightarrow x < u$ (cf. [9]).

4.3 Remark. Let h be any element from a partially ordered quasigroup Q. To see that the condition (IP) holds, it is sufficient to show that for all elements $x, y, z \in Q$ such that h, x are non-comparable, y, z are non-comparable, $h < y, z$ and $x < y, z$ there exists $c \in Q$ such that $h, x \leq c \leq y, z$.

Using similar methods as in [6] (by 4.3, the group unit can be replaced by any $h \in Q$) we can prove both following lemmas.

4.4 Lemma (cf. [6, Lemma 2.1])**.** *Let I be a well-ordered set, $\{Q_i : i \in I\}$ a family of partially ordered quasigroups. If $Q = \Gamma_{i \in I} Q_i$ satisfies* (IP)*, then each Q_i satisfies* (IP)*, too.*

4.5 Lemma (cf. [6, Lemma 2.2])**.** *Let I be a well-ordered set, $\{Q_i : i \in I\}$ a family of partially ordered quasigroups. Let $Q = \Gamma_{i \in I} Q_i$. If $h \leq u, v$, $a \leq u, v$, $a \parallel h$, $u \parallel v$ for some $h, a, u, v \in Q$, then there exists an index $i \in I$ such that $a(i) < u(i), v(i)$, $h(i) < u(i), v(i)$ and $h(j) = a(j) = u(j) = v(j)$ for all $j \in I$, $j < i$.*

By an *antilattice* we mean such a po-quasigroup, in which only pairs of comparable elements may have a greatest lower and a least upper bound. Choose any element $h \in Q$. To verify that a partially ordered quasigroup Q is an antilattice, it is sufficient to show that if $a \in Q$, $a \parallel h$, then a, h do not have a least upper bound.

A partially ordered quasigroup Q will be said to be *dense* if, whenever $a, b \in Q$, $a < b$, there exists $c \in Q$ with $a < c < b$. And again, to see that Q is dense, it is sufficient to show that for any chosen (fixed) element $h \in Q$ and all $b \in Q$ such that $h < b$, there exists $c \in Q$ with $h < c < b$.

The following theorem generalizes Theorem 2.3 in [6] which was formulated for Riesz groups.

4.6 Theorem. *Let I be a well-ordered set and $\{Q_i : i \in I\}$ a family of partially ordered quasigroups such that each Q_i contains more than one element. Let $Q = \Gamma_{i\in I} Q_i$. Then Q satisfies* (IP) *if and only if all Q_i satisfy* (IP) *and for each couple (j,k) of elements of I such that k covers j it is true that if Q_k is not directed and Q_j is non-trivially ordered, then the quasigroup Q_j is a dense antilattice.*

Proof: Let Q satisfy (IP). Let j,k be such elements of I, that k covers j and let Q_k be not directed, Q_j be non-trivially ordered. By 4.1 there exist $e_k, t_k, r_k \in Q_k$ such that $U(e_k,t_k) = \emptyset$ and $L(e_k,r_k) = \emptyset$. Take any element $e \in Q$ with $e(k) = e_k$. To prove that Q_j is dense it is sufficient to verify that for each $g_j \in Q_j$, $e(j) < g_j$, there exists $h_j \in Q_j$ such that $e(j) < h_j < g_j$. Define elements $a,u,v \in Q$ by $a(j) = e(j)$, $u(j) = v(j) = g_j$, $a(k) = t_k$, $u(k) = e_k$, $v(k) = r_k$ and $a(l) = u(l) = v(l) = e(l)$ for all $l \in I - \{j,k\}$. We have $e < u,v$, $a < u,v$, $a \parallel e$, $u \parallel v$. By assumption there exists $p \in Q$ such that $e,a < p < u,v$. Evidently $p(i) = e(i)$ for all $i < j$, $e(j) \le p(j) \le g_j$. If $p(j) = e(j)$, then $p(k) \ge e_k, t_k$, a contradiction. On the other hand, if $p(j) = g_j$, then $p(k) \le e_k, r_k$, which is again a contradiction. So we have $e(j) < p(j) < g_j$ which proves the density of Q_j.

The rest of the proof can be performed by using the same methods as in [6]. □

4.7 Corollary (cf. [6]). *Let I be a well-ordered set with the least element i_0 and let $\{Q_i : i \in I\}$ be a family of partially ordered quasigroups such that each Q_i contains more than one element. Then $Q = \Gamma_{i\in I} Q_i$ is a Riesz quasigroup if and only if the following conditions are satisfied:*

(i) *Q_{i_0} is a directed quasigroup;*
(ii) *all Q_i's satisfy* (IP);
(iii) *if $j,k \in I$, k covers j, Q_k is not directed, Q_j is non-trivially ordered, then Q_j is a dense antilattice.*

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